Equigeodesics on full, G_2 and a rank-three condiction on flag manifolds

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Abstract

This paper provides a characterization and examples of homogeneous geodesics on full G/T and G_2 flag manifolds. We discuss for generalized root systems the property of sum-zero triple of T-roots and give several applications of this result.

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1 Introduction

An important class of homogeneous manifolds are the orbits of the adjoint action of a semisimple compact Lie group, called *generalized flag manifolds*. Such manifolds can be described by a quotient $\mathbb{F} = G/C(T)$, where $C(T)$ is the centralizer of a torus T of the Lie group G. If $C(T) = T$ then $\mathbb{F} = G/T$ is called *full flag manifold*.

These manifolds were studied by many authors by the 50's, with focus on its topological properties [4]. There are also many recent papers related to the G-invariant geometry in flag manifolds, for instance [11], [20], [1], [8] and [12].

This paper deals with two classical subjects in Riemannian geometry: geodesics and G-invariant geometries.

Let G/K be a homogeneous manifold with origin $o = eK$ (trivial coset) and g be a G-invariant metric. A geodesic $\gamma(t)$ on G/K through the origin o is called *homogeneous* if it is the orbit of a 1-parameter subgroup of G, that is,

$$\gamma(t) = (\exp tX) \cdot o,$$

where $X \in \mathfrak{g}$. The vector X is called a geodesic vector.

In [8] it was introduced the notion of *homogeneous equigeodesics*. An *homogeneous equigeodesic* is an homogeneous curve γ which is *geodesic with respect to any G-invariant metric*. It was obtained condictions for geometrical flag manifolds (i.e. of type A_l) to admit homogeneous equigeodesics. All such condictions were described in terms of equigeodesic vectors.

In this paper we provide a characterization of all homogeneous equigeodesics in any full flag manifold G/T. Such characterization, given in terms of the equigeodesic vectors (see Section 5 for further details). Our first result is

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Theorem A: *Let $\mathbb{F} = G/T$ with T a maximal torus on G and $X \in T_o\mathbb{F}$. If $X = X_{\alpha_1} + \ldots + X_{\alpha_n}$, with $X_{\alpha_i} \in \mathfrak{u}_{\alpha_i}$ where $\mathfrak{u}_{\alpha_i}$ is the root space associated to the root α_i then, the curve $\gamma(t) = (\exp tX) \cdot o$ is an equigeodesic if, and only if, $\alpha_i \pm \alpha_j$ are not roots, for any $i, j \in \{1, \ldots, n\}$.*

After Theorem A**,** we studied homogeneous equigeodesics in flag manifolds of type G_2 obtaining its classification. According to [6] these manifolds are classified in three types: $G_2(\alpha_1) = G_2/U(2)$, where $U(2)$ is represented by the long root; $G_2(\alpha_2) = G_2/U(2)$, where $U(2)$ is represented by the short root; and G_2/T, where T is a maximal torus of G_2. According Theorem A, we only have to discuss the cases $G_2(\alpha_1)$ and $G_2(\alpha_2)$.

The tangent space at origin o (trivial coset) of $G_2(\alpha_2)$ can be written as

$$T_oG_2(\alpha_2) = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{2\alpha_1}$$

where $\mathfrak{m}_{\alpha_1}, \mathfrak{m}_{2\alpha_1}$ are the irreducible submodules of the isotropic representation. Thus the next two results classify all the *equigeodesics* on $G_2(\alpha_1)$ and $G_2(\alpha_2)$.

Theorem B: *Let $G_2(\alpha_2) = G_2/U(2)$. Then X is an* equigeodesic vector *if, and only if, $X \in \mathfrak{m}_{\alpha_1}$ or $X \in \mathfrak{m}_{2\alpha_1}$.*

On the other hand for the flag manifolds $G_2(\alpha_1)$ we prove the following

Theorem C: *A vector $X \in T_oG_2(\alpha_1)$ is an equigeodesic vector iff the coefficients of X are solutions of a non-linear algebraic system of equations. Such system of equations can be solved explicitly.*

In section 5 we also provide several examples of homogeneous equigeodesics in any class of full and G_2 type of flag manifolds: G/T, $G_2(\alpha_1)$, $G_2(\alpha_2)$.

One key point in the understanding of the invariant Hermitian geometry of flags is the study of the behavior of triples of roots (the relevance of this fact it was first noticed in [11]). The concept of sum-zero triple for root systems it was introduced in [9] and is naturally associated to the study of (1,2)-symplectic metrics on flags.

In this paper following [1] we generalize the notion of root systems for an arbitrary flag manifold $\mathbb{F} = G/K$ and call it by a system of T-roots R_T. It is defined as the restriction of the root system R of the corresponding Lie algebra $\mathfrak{g}$ to the center $\mathfrak{t}$ of the (stability) subalgebra $\mathfrak{k}$ of K.

The following result is useful in order to determine the set of T-roots and is connected to properties of Einstein metrics on flag manifolds.

Theorem D: *Let $M = G/K$ and R_T a corresponding set of T-roots. If R_T contains more than one positive T-root then every T-root belongs to some T-zero sum triple.*

In a forthcoming paper [10] we will apply this result in order to obtain a description of Hermitian classes in terms of T-roots. In [20] all the invariant Hermitian structures were classified on full flag manifolds. Among all these metrics it is natural to determine the ones that are Einstein.

We can related a result in ([24], Corollary 1.5) concerning to the normal metric with theorem D and a result we have derived connecting Einstein metrics with the dimensions of the modules $\mathfrak{m}_i$.

Corollary E: *Let $\mathbb{F} = G/K$ and $T_oM = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ then $R_T = \{\pm\zeta, \pm 2\zeta\}$ where $\zeta \in R_T - \{0\}$. Furthermore, if $\dim\, \mathfrak{m}_1 \neq \dim\, \mathfrak{m}_2$ then any invariant Eintein metric on $\mathbb{F}$ satisfies $\lambda_1 \neq \lambda_2$.*

We, just for completeness, derive the Einstein equations for G_2/T and describe explicitly the invariant Kähler-Einstein metric correponding to each invariant complex structures on G_2/T.

The paper is organized in the following format. In the Sections 2 and 3 we summarize some results about the geometry of the flag manifolds, describe the isotropy representation, T-roots, invariant metrics and the invariant almost-complex structures. In the Section 4 we present the classification of the flag manifolds of type G_2. In the Section 5 we prove a characterization of equigeodesic vector on full and G_2 flag manifolds

and give several examples of such curves. Finally, in Section 6, we present the result about the zero sum triple of T-roots, with applications in the study of the invariant Einstein metrics.

2 Flag manifolds

In this section we briefly review some basic facts on the structure of homogeneous spaces, flag manifolds and describe the T-root system.

I. Homogeneous spaces. Consider the homogeneous manifold $M = G/K$ with G a compact semi-simple Lie group and K a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. The Cartan-Killing form $\langle , \rangle$ is nondegenerate and negative definite in $\mathfrak{g}$, thus giving rise to the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant. We may identify $\mathfrak{m}$ with the tangent space T_oM at $o = eK$. The isotropy representation of a reductive homogeneous space is equivalent to the homomorphism $j : K \to GL(T_oM)$ given by $j(k) = \mathrm{Ad}(k)\big|_{\mathfrak{m}}$.

II. Generalized flag manifolds. A homogeneous space $\mathbb{F} = G/K$ is called a generalized flag manifold if G is simple Lie group and the isotropy group K is the centralizer of a one-parametrer subgroup of G, $\exp tw$ ($w \in \mathfrak{g}$). Equivalently, $\mathbb{F}$ is an adjoint orbit $\mathrm{Ad}(G)w$, where $w \in \mathfrak{g}$. The generalized flag manifolds (also refereed to as a Kählerian C-spaces) have been classified in [6],[22].

Here the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ has a more complete description. Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of the complexification $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{k}$, which is also a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let R and R_K be the root systems of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$, respectively, and $R_M = R \backslash R_K$ be the set of complementary roots. We have the Cartan decompositions

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha, \qquad \mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ denotes the root space associated to root α and $\mathfrak{m}^{\mathbb{C}}$ is isomorphic to $(T_o\mathbb{F})^{\mathbb{C}}$ and $\mathfrak{h} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}$.

We fix a Weyl basis in $\mathfrak{m}^{\mathbb{C}}$, namely, elements $E_\alpha \in \mathfrak{g}_\alpha$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$ and $[E_\alpha, E_\beta] = m_{\alpha,\beta} E_{\alpha+\beta}$, with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{\alpha,\beta} = -m_{\beta,\alpha}$, $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$ and $m_{\alpha,\beta} = 0$ if, and only if, $\alpha + \beta$ is not a root. The corresponding *real* Weyl basis in $\mathfrak{m}$ consists of the vectors $A_\alpha = E_\alpha - E_{-\alpha}$, $S_\alpha = E_\alpha + E_{-\alpha}$ and $\mathfrak{u}_\alpha = \mathrm{span}_{\mathbb{R}}\{A_\alpha, iS_\alpha\}$, where $\alpha \in R^+$, the set of positive roots.

The real tangent space $T_o\mathbb{F}$ is naturally identified with

$$\mathfrak{m} = \bigoplus_{\alpha \in R_M^+} \mathfrak{u}_\alpha.$$

Some of the spaces $\mathfrak{u}_\alpha$ are not $\mathrm{Ad}(K)$-modules, unless $\mathbb{F}$ is a full flag manifold. To get the *irreducible $Ad(K)$-modules*, we proceed as in [2]. Let

$$\mathfrak{t} = Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h} = \{X \in \mathfrak{h} : \phi(x) = 0 \ \forall \phi \in R_K\}.$$

If $\mathfrak{h}^*$ and $\mathfrak{t}^*$ are the dual space of $\mathfrak{h}$ and $\mathfrak{t}$ respectively, we consider the restriction map

$$\kappa : \mathfrak{h}^* \to \mathfrak{t}^*, \qquad \kappa(\alpha) = \alpha|_{\mathfrak{t}} \tag{1}$$

and set $R_T = \kappa(R_M)$. The elements on R_T are called T-roots. The irreducible $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant sub-modules of $\mathfrak{m}^{\mathbb{C}}$, and the corresponding irreducible sub-modules for the $\mathrm{ad}(\mathfrak{k})$-module $\mathfrak{m}$, are given by

$$\mathfrak{m}^{\mathbb{C}}_\xi = \sum_{\kappa(\alpha)=\xi} \mathfrak{g}_\alpha \quad (\xi \in R_T), \qquad \mathfrak{m}_\eta = \sum_{\kappa(\alpha)=\eta} \mathfrak{u}_\alpha \quad (\eta \in R_T^+ = \kappa(R_M^+)).$$

Hence we have the direct sum of complex and real irreducible modules,

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\eta \in R_T} \mathfrak{m}^{\mathbb{C}}_\eta, \qquad \mathfrak{m} = \sum_{\eta \in R_T^+} \mathfrak{m}_\eta.$$

3 Invariant metrics and iacs

An invariant metric g on $\mathbb{F}$ is uniquely determined by a scalar product B on $\mathfrak{m}$ of the form

$$B(\cdot\,,\cdot) = -\langle\Lambda\cdot\,,\cdot\rangle = \lambda_1(-\langle\cdot\,,\cdot\rangle)|_{\mathfrak{m}_1} + \ldots + \lambda_s(-\langle\cdot\,,\cdot\rangle)|_{\mathfrak{m}_s},$$

where the linear map $\Lambda : \mathfrak{m} \to \mathfrak{m}$ is symmetric, positive-definite with respect to the Cartan-Killing form, $\lambda_i > 0$ and $\mathfrak{m}_i$ are the irreducible $\mathrm{Ad}(K)$-sub-modules. Each $\mathfrak{m}_i$ is an eigenspace of Λ corresponding to the eigenvalue λ_i. In particular, the vectors A_α, S_α of the real Weyl basis are eigenvectors of Λ correponding to the same eigenvalue λ_α. We abuse of notation and say that Λ itself is an invariant metric.

The inner product B admits a natural extension to a symmetric bilinear form on the complexification $\mathfrak{m}^{\mathbb{C}}$ of $\mathfrak{m}$. We do not change notation for these objects in $\mathfrak{m}$ and $\mathfrak{m}^{\mathbb{C}}$ either for the bilinear form B or for the corresponding complexified map Λ.

It is well known that an G-invariant almost complex structure (abbreviated iacs) on $\mathbb{F}$ is completely determined by its value $J\colon \mathfrak{m} \to \mathfrak{m}$ in the tangent space at the origin. The linear endomorphism J satisfies $J^2 = -1$ and $Ad(K)J = JAd(K)$. We will also denote by J its complexification to $\mathfrak{m}^{\mathbb{C}}$. The eigenvalues of J are $\pm i$ and the corresponding eigenvectors are denoted by $T_o^{(1,0)}\mathbb{F} = \{X \in T_o\mathbb{F} : JX = iX\}$ and $T_o^{(0,1)}\mathbb{F} = \{X \in T_o\mathbb{F} : JX = -iX\}$. Thus we have the following decomposition of the tangent complex space at the origin $\mathfrak{m}^{\mathbb{C}} = T_o^{(1,0)}\mathbb{F} \oplus T_o^{(0,1)}\mathbb{F}$. The invariance of J entails that $J(\mathfrak{g}_\alpha^{\mathbb{C}}) = \mathfrak{g}_\alpha^{\mathbb{C}}$ for all $\alpha \in R$. Then $JE_\alpha = i\varepsilon_\alpha E_\alpha$, with $\varepsilon_\alpha = \pm 1$.

As $A_\alpha = E_\alpha - E_{-\alpha}$ and $S_\alpha = E_\alpha + E_{-\alpha}$ we obtain that $iA_\alpha = -iA_{-\alpha}$ and $S_\alpha = S_{-\alpha}$. Note that $E_{-\alpha} = \frac{1}{2}(i(iA_\alpha) + S_\alpha)$, then

$$i\varepsilon_{-\alpha}\frac{1}{2}(i(iA_\alpha) + S_\alpha) = i\varepsilon_{-\alpha}E_{-\alpha} = JE_{-\alpha} = \frac{1}{2}(iJ(iA_\alpha) + J(S_\alpha)).$$

Comparing the reals and imaginary terms in the left side of the first equation and in the right side of the third equation, we obtain $J(iA_\alpha) = \varepsilon_{-\alpha}S_\alpha$ and $J(S_\alpha) = -\varepsilon_{-\alpha}(iA_\alpha)$, then

$$\varepsilon_{-\alpha}S_\alpha = J(iA_\alpha) = -J(iA_{-\alpha}) = -\varepsilon_\alpha S_{-\alpha},$$

so $\varepsilon_\alpha = -\varepsilon_{-\alpha}$, with $\alpha \in R$.

The irreducible $ad(\mathfrak{k}^{\mathbb{C}})$-modules are $\mathfrak{m}_i^{\mathbb{C}}$ invariant by J, that is, $J\mathfrak{m}_i^{\mathbb{C}} = \mathfrak{m}_i^{\mathbb{C}}$. Then using the Schur's Lemma we get

$$J = i\varepsilon_1 Id|_{\mathfrak{m}_1^{\mathbb{C}}} \oplus \cdots \oplus i\varepsilon_r Id|_{\mathfrak{m}_r^{\mathbb{C}}}.$$

Moreover, if δ is a T-root we have $\varepsilon_\delta = \varepsilon_\alpha = -\varepsilon_{-\alpha} = -\varepsilon_{-\delta}$ where α is any root in R such that $k(\alpha) = \delta$. Thus we obtain

Proposition 1. *Let $\mathbb{F}$ be a flag manifold and R_T the correspondent set of T-roots. Then any iacs J on $\mathbb{F}$ is completely determined by a set of sign $\{\varepsilon_\delta, \delta \in R_T\}$ ($\varepsilon_\delta = \pm 1$) satisfying $\varepsilon_\delta = -\varepsilon_{-\delta}$ with $\delta \in R_T$. In particular, J is determined by exactly $|R_T^+|$ signs.*

An iacs J is integrable if, and only if, it is torsion free, that is,

$$[JX, JY] = [X, Y] + J[X, JY] + J[JX, Y] \qquad X, Y \in \mathfrak{m},\ \text{(see, for instance, [16]).}$$

4 Flag manifolds of G_2 type.

We recall some basics facts about the Lie algebra of G_2. We can realize the Lie algebra of G_2 as the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C}) \oplus \mathbb{C}^3 \oplus \left(\mathbb{C}^3\right)^*$. A Cartan subalgebra $\mathfrak{h}$ of diagonal matrices on $\mathfrak{sl}(3,\mathbb{C})$ is also a Cartan subalgebra on $\mathfrak{g}$.

Consider the linear functional ε_i de $\mathfrak{h}$ defined by: $\varepsilon_i\colon diag\{a_1,a_2,a_3\} \mapsto a_i$. A basis for the root system relative to $(\mathfrak{g},\mathfrak{h})$ is given by $\Sigma = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2\}$. The corresponding positive roots are $R^+ = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2,\ \alpha_1 + \alpha_2 = \varepsilon_1,\ \alpha_1 + 2\alpha_2 = -\varepsilon_3,\ \alpha_1 + 3\alpha_2 = \varepsilon_2 - \varepsilon_3,\ 2\alpha_1 + 3\alpha_2 = \varepsilon_1 - \varepsilon_3\}$. The Cartan-Killing form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ is given by:

$$(2\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2) = (\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_2) = (\alpha_1,\alpha_1) = \frac{1}{4} \quad \text{(long roots)}$$
$$(\alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_2) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) = (\alpha_2,\alpha_2) = \frac{1}{12} \quad \text{(short roots)}\ .$$

According to [6] there are only three non-equivalent classes of G_2 flag manifolds.

The following table list these manifolds.

Flag manifold	$\mathfrak{m} = \oplus_{i=1}^{t}\mathfrak{m}_i$
$G_2(\alpha_1) = G_2/U(2)$, where $U(2)$ is represented by the long root	$t = 3$
$G_2(\alpha_2) = G_2/U(2)$, where $U(2)$ is represented by the short root	$t = 2$
G_2/T, where T is a maximal torus of G_2	$t = 6$

The second column represents the number of the irreducible non-equivalent submodules of the isotropic representation.

5 Homogeneous geodesics and equigeodesics

In this section we give a characterization of homogeneous equigeodesics in full flag manifolds and flag manifolds of G_2. We start with an definition.

Definition 1. *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. A geodesic $\gamma(t)$ on M through the origin o is called homogeneous if it is the orbit of a 1-parameter subgroup of G, that is,*

$$\gamma(t) = (\exp tX)\cdot o,$$

where $X \in \mathfrak{g}$. The vector X is called a geodesic vector.

A useful result of Kowalski and Vanhecke [18] gives an algebraic characterization of the geodesic vectors.

Theorem 1. *If g is a G-invariant metric, a vector $X \in \mathfrak{g}\setminus\{0\}$ is a geodesic vector if, and only if,*

$$g(X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}) = 0, \tag{2}$$

for all $Z \in \mathfrak{m}$.

An important class of the homogeneous manifolds are the *g.o. manifolds* (geodesic orbit manifold). We say that a homogeneous manifolds is a g.o manifold if it admits a invariant Riemannian metric such that *all* geodesics are *homogeneous*. Examples of these manifold are the homogeneous space equipped with the normal metric and the symmetric spaces.

It is well know that neither full flag manifolds G/T nor flag manifolds of G_2 admits a left invariant metric (not homotetic to normal metric) such that all geodesics are homogeneous (see [1]).

On other hand, every homogeneous manifold admits at least one homogeneous geodesics, see [17]. In case of the group G is semi-simple we have the following existence result:

Theorem 2 ([17])**.** *If G is semi-simple then $M = G/K$ admits at least $m = dim(M)$ mutually orthogonal homogeneous geodesics through the origin o.*

An *homogeneous equigeodesic* is an homogeneous curve that is geodesic with respect to any invariant metric. In [8] it was proved that any flag manifold admits equigeodesics. The following algebraic characterization of equigeodesics is given in terms of the *equigeodesic vectors*, that is, vectors $X \in \mathfrak{m}$ such that the orbit $\gamma(t) = (\exp tX) \cdot o$ is an homogeneous equigeodesic.

Proposition 2 ([8]). *Let $\mathbb{F}$ be a flag manifold, with $\mathfrak{m}$ isomorphic to T_oF. A vector $X \in \mathfrak{m}$ is equigeodesic vector if, and only if,*

$$[X, \Lambda X]_{\mathfrak{m}} = 0, \tag{3}$$

for any invariant metric Λ.

Proof: Let g be the metric associated with Λ. For $X, Y \in \mathfrak{m}$ we have

$$g(X, [X,Y]_{\mathfrak{m}}) \quad = \quad -\langle \Lambda X, [X,Y]_{\mathfrak{m}} \rangle = -\langle \Lambda X, [X,Y] \rangle = -\langle [X, \Lambda X], Y \rangle ,$$

since the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is $\langle , \rangle$-orthogonal and the Killing form is $\mathrm{Ad}(G)$-invariant, i.e., $\mathrm{ad}(X)$ is skew-Hermitian with respect to $<,>$. Therefore X is equigeodesic iff $[X, \Lambda X]_{\mathfrak{m}} = 0$ for any invariant scalar product Λ. □

We will now give a full characterization of the equigeodesics or equivalently equigeodesic vectors on *any* full flag manifold G/T, where G is a compact, connected and simple Lie group and T is a maximal torus on G. We recall that the irreducible submodules of the isotropy representation in full flag manifolds coincides with $\mathfrak{u}_\alpha = span_{\mathbb{R}}\{A_\alpha, iS_\alpha\}, \alpha \in R^+$.

Theorem 3. *Let G be a compact, connected and simple Lie group with Lie algebra $\mathfrak{g}$, T a maximal torus in G and G/T the corresponding full flag manifold. Let $X \in \mathfrak{u}_\alpha$, $Y \in \mathfrak{u}_\beta$ be nonzero vectors. Then $X + Y \in \mathfrak{m}$ is an equigeodesic vectors if, and only if, $\alpha \pm \beta$ are not roots.*

Proof: Let $\{E_\alpha\}_{\alpha \in R}$ be the Weyl´s basis of $\mathfrak{g}^{\mathbb{C}}$ the complexification of the real simple algebra $\mathfrak{g}$, and set $X = a_1 A_\alpha + b_1 i S_\alpha \in \mathfrak{u}_\alpha, Y = a_2 A_\beta + b_2 i S_\beta \in \mathfrak{u}_\beta$, with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Using the Weyl´s basis we can write $X = cE_\alpha - \overline{c}E_{-\alpha}$ and $Y = dE_\beta - \overline{d}E_{-\beta}$, where $c = a_1 + ib_1$ and $d = a_2 + ib_2$. Then,

$$\begin{aligned} [X+Y, \Lambda(X+Y)]_{\mathfrak{m}} &= [cE_\alpha - \overline{c}E_{-\alpha} + dE_\beta - \overline{d}E_{-\beta},\, c\lambda_\alpha E_\alpha - \overline{c}\lambda_\alpha E_{-\alpha} + d\lambda_\beta E_\beta - \overline{d}\lambda_\beta E_{-\beta}] \\ &= (\lambda_\beta - \lambda_\alpha) cd\, m_{\alpha,\beta} E_{\alpha+\beta} - (\lambda_\beta - \lambda_\alpha) c\overline{d}\, m_{\alpha,-\beta} E_{\alpha-\beta} \\ &\quad -(\lambda_\beta - \lambda_\alpha)\overline{c}d\, m_{-\alpha,\beta} E_{-\alpha+\beta} + (\lambda_\beta - \lambda_\alpha)\overline{c}\overline{d}\lambda_\beta\, m_{-\alpha,-\beta} E_{-\alpha-\beta}. \end{aligned} \tag{4}$$

Suppose that $X + Y$ is equigeodesic. Then $[X + Y, \Lambda(X + Y)]_{\mathfrak{m}} = 0$ for any invariant metric Λ and from equation (4) we have $m_{\alpha,\beta} = -m_{-\alpha,-\beta} = 0$ e $m_{\alpha,-\beta} = -m_{-\alpha,\beta} = 0$ because c and d are nonzero and therefore $\alpha \pm \beta$ are not roots.

On other hand, suppose that $\alpha \pm \beta$ are not roots. Then $m_{\alpha,\beta} = -m_{-\alpha,-\beta} = 0$ and $m_{\alpha,-\beta} = -m_{-\alpha,\beta} = 0$ and from (4) we have $[X + Y, \Lambda(X + Y)]_{\mathfrak{m}} = 0$ for any invariant metric Λ and $X + Y$ is an equigeodesic vector. □

Corollary 1. *With the hypothesis from Theorem 3, let $X = X_{\alpha_1} + \ldots + X_{\alpha_r}$ such that $X_{\alpha_i} \in \mathfrak{u}_{\alpha_i}$ for all i. Then X is an equigeodesic if, and only if, $\alpha_p \pm \alpha_q$ are not roots for every $p, q \in \{1, \ldots, r\}$.*

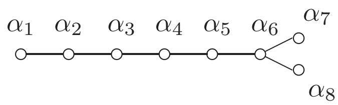
Proof: Apply Theorem above and the linearity of the Lie bracket. □

Example 1 ([8]). *Consider the Lie algebra $A_l = \mathfrak{sl}(l, \mathbb{C})$. The Cartan sub-algebra of $\mathfrak{sl}(n, \mathbb{C})$ can be identified with $\mathfrak{h} = \{diag(\varepsilon_1, \ldots, \varepsilon_n); \varepsilon_i \in \mathbb{C}, \sum \varepsilon_i = 0\}$.*

The root system of the Lie algebra of $\mathfrak{sl}(n)$ has the form $R = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j : i \neq j\}$ and the subset of positive roots is $R^+ = \{\alpha_{ij} : i < j\}$, see [14]. Therefore, $\alpha_{ij} \pm \alpha_{pq}$ is not a root if, and only if, i, j, p, q are all distinct.

Example 2. *Consider the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$, R beeing an associated root system, and Σ a simple root system. Two simple roots are said to be* orthogonal *if they are not joined in the Dynkin diagram. If α_1 and α_2 are two orthogonal simple roots then $\alpha_1 \pm \alpha_2$ are not roots, see [14].*

For example, on the full flag manifold $SO(16)/T$ we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(16, \mathbb{C})$ (a Lie algebra of type D_l) and the associated Dynkin diagram is given by



Hence, any element in the set $\mathfrak{u}_{\alpha_1} \oplus \mathfrak{u}_{\alpha_3}$ is an equigeodesic vector since $\alpha_1 \pm \alpha_3$ are not roots. In the same way, any element in the set $\mathfrak{u}_{\alpha_2} \oplus \mathfrak{u}_{\alpha_4} \oplus \mathfrak{u}_{\alpha_7}$ is equigeodesic vector.

In a completely similar way we find equigeodesic vectors in any full flag manifold.

Example 3. *Let G_2/T be the full flag manifold and consider the roots $\alpha = \alpha_2$ and $\beta = 2\alpha_1 + 3\alpha_2$. Then any element in the set $\mathfrak{u}_\alpha \oplus \mathfrak{u}_\beta$ is an equigeodesic vector.*

We will now give a complete description of the equigeodesic vectors in flag manifolds of G_2.

a) Let $G_2(\alpha_2) = G_2/U(2)$ represented by the short root. In this case we have $R_K = \{\alpha_2\}$ and $R_M = \{\alpha_1\}$. The isotropic representation have two irreducible submodules and the tangent space at o splits as $\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{2\alpha_1}$, where $\mathfrak{m}_{\alpha_1} = \mathfrak{u}_{\alpha_1} \oplus \mathfrak{u}_{\alpha_1+\alpha_2} \oplus \mathfrak{u}_{\alpha_1+2\alpha_2} \oplus \mathfrak{u}_{\alpha_1+3\alpha_2}$ and $\mathfrak{m}_{2\alpha_1} = \mathfrak{u}_{2\alpha_1+3\alpha_2}$.

Lemma 1. *Let $\mathfrak{m}_{\alpha_1}, \mathfrak{m}_{\alpha_2}$ be the irreducibles submodules of the isotropy representation of the flag manifold $G_2(\alpha_2)$. Then $[\mathfrak{m}_{\alpha_1}, \mathfrak{m}_{2\alpha_1}] \subset \mathfrak{m}_{\alpha_1}$.*

Proof: Let $\{E_\alpha\}_{\alpha \in R_M}$ be a Weyl's basis of $\mathfrak{g}_2$ and let $X \in \mathfrak{m}_{\alpha_1}$ and $Y \in \mathfrak{m}_{2\alpha_1}$. Writing

$$X = a_1 E_{\alpha_1} + a_2 E_{\alpha_1+\alpha_2} + a_3 E_{\alpha_1+2\alpha_2} + a_4 E_{\alpha_1+3\alpha_2} + b_1 E_{-\alpha_1} + b_2 E_{-(\alpha_1+\alpha_2)} + b_3 E_{-(\alpha_1+2\alpha_2)} + b_4 E_{-(\alpha_1+3\alpha_2)},$$

$$Y = c_1 E_{2\alpha_1+3\alpha_2} + c_2 E_{-(2\alpha_1+3\alpha_2)},$$

with $b_i = -\overline{a_i}$ e $c_2 = -\overline{c_1}$, we have

$$\begin{aligned} [X, Y] &= a_1 c_1 E_{-(\alpha_1+3\alpha_2)} + a_2 c_2 E_{-(\alpha_1+2\alpha_2)} + a_3 c_2 E_{-(\alpha_1+\alpha_2)} + a_4 c_2 E_{-\alpha_1} \\ &\quad - b_1 c_1 E_{\alpha_1+3\alpha_2} - b_2 c_1 E_{\alpha_1+2\alpha_2} - b_3 c_1 E_{\alpha_1+\alpha_2} - b_4 c_1 E_{\alpha_1} \end{aligned} \tag{5}$$

Therefore $[X, Y] \in \mathfrak{m}_{2\alpha}$. □

Proposition 3. *Let $G_2/U(2)$ be the flag manifold represented by the short root. Let $X \in \mathfrak{m} = T_oF$ be a nonzero vector. Then X is equigeodesic if, and only if, $X \in \mathfrak{m}_{\alpha_1}$ or $X \in \mathfrak{m}_{2\alpha_1}$.*

Proof: Writing $X = X_{\alpha_1} + X_{2\alpha_1}$ with $X_{\alpha_1} \in \mathfrak{m}_{\alpha_1}$ and $X_{2\alpha_1} \in \mathfrak{m}_{2\alpha_1}$, we have

$$\begin{aligned} [X, \Lambda X] &= [X_{\alpha_1} + X_{2\alpha_1}, \lambda_1 X_{\alpha_1} + \lambda_2 X_{2\alpha_1}] \\ &= \lambda_1 [X_{\alpha_1}, X_{\alpha_1}] + \lambda_2 [X_{\alpha_1}, X_{2\alpha_1}] + \lambda_1 [X_{2\alpha_1}, X_{\alpha_1}] + \lambda_2 [X_{2\alpha_1}, X_{2\alpha_1}] \\ &= (\lambda_2 - \lambda_1)[X_{\alpha_1}, X_{2\alpha_1}]. \end{aligned} \tag{6}$$

If X is equigeodesic then $(\lambda_2 - \lambda_1)[X_{\alpha_1}, X_{2\alpha_1}] = 0$ for any $\lambda_1 > 0, \lambda_2 > 0$ and therefore $[X_{\alpha_1}, X_{2\alpha_1}] = 0$. According the previous lemma, $[X_{\alpha_1}, X_{2\alpha_1}] = 0$ if, and only if $X_{\alpha_1} = 0$ or $X_{2\alpha_1} = 0$.

On other hand, if $X \in \mathfrak{m}_{\alpha_1}$ then $\Lambda X = \lambda_1 X$ for any invariant metric Λ and $[X, \Lambda X]$=0 for any Λ and X is equigeodesic vector. Analogously for $X \in \mathfrak{m}_{2\alpha_1}$. □

b) Consider now $G_2(\alpha_1) = G_2/U(2)$ represented by the long root. In this case we have $R_K = \{\alpha_1\}$ and $R_M = \{\alpha_2\}$. The isotropic representation have three irreducible submodules and the tangent space at o slipts as $\mathfrak{m} = \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{2\alpha_2} \oplus \mathfrak{m}_{3\alpha_2}$, where $\mathfrak{m}_{\alpha_2} = \mathfrak{u}_{\alpha_2} \oplus \mathfrak{u}_{\alpha_1+\alpha_2}$, $\mathfrak{m}_{2\alpha_2} = \mathfrak{u}_{\alpha_1+2\alpha_2}$ and $\mathfrak{m}_{3\alpha_2} = \mathfrak{u}_{\alpha_1+3\alpha_2} \oplus \mathfrak{u}_{2\alpha_1+3\alpha_2}$. As in Lemma 1 we prove

Lemma 2. *The following inclusions hold:*

1. $[\mathfrak{m}_{\alpha_2}, \mathfrak{m}_{2\alpha_2}] \subset \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{3\alpha_2}$;
2. $[\mathfrak{m}_{\alpha_2}, \mathfrak{m}_{3\alpha_2}] \subset \mathfrak{m}_{2\alpha_2}$;
3. $[\mathfrak{m}_{2\alpha_2}, \mathfrak{m}_{3\alpha_2}] \subset \mathfrak{m}_{\alpha_2}$.

Proof: Let E_α be a Weyl's basis of $\mathfrak{g}_2$. Writting

$$\begin{array}{rcl} X &=& a_1 E_{\alpha_2} + a_2 E_{\alpha_1+\alpha_2} + b_1 E_{-\alpha_2} + b_2 E_{-(\alpha_1+\alpha_2)} \in \mathfrak{m}_{\alpha_2}, \\ Y &=& c_1 E_{\alpha_1+2\alpha_2} + c_2 E_{-(\alpha_1+2\alpha_2)} \in \mathfrak{m}_{2\alpha_2}, \\ Z &=& d_1 E_{\alpha_1+3\alpha_2} + d_2 E_{2\alpha_1+3\alpha_2} + e_1 E_{-(\alpha_1+3\alpha_2)} + e_2 E_{-(2\alpha_1+3\alpha_2)} \in \mathfrak{m}_{3\alpha_2}, \end{array} \tag{7}$$

with $b_i = -\overline{a_i}, c_2 = -\overline{c_1}$ and $e_i = -\overline{d_i}$ we have

$$\begin{array}{rcl} 1)\,[X,Y] &=& a_1c_1E_{\alpha_1+3\alpha_2} + a_1c_2E_{-(\alpha_1+\alpha_2)} + a_2c_1E_{2\alpha_1+3\alpha_2} + a_2c_2E_{-\alpha_2} \\ &-& b_1c_1E_{\alpha_1+\alpha_2} - b_1c_2E_{-(\alpha_1+3\alpha_2)} - b_2c_1E_{\alpha_2} - b_2c_2E_{-(2\alpha_1+3\alpha_2)} \subset \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{3\alpha_2}; \end{array}$$

$$\begin{array}{rcl} 2)\,[X,Z] &=& a_1e_1E_{-(\alpha_1+2\alpha_2)} + a_2e_2E_{-(\alpha_1+2\alpha_2)} - b_1d_1E_{\alpha_1+2\alpha_2} - b_2d_2E_{\alpha_1+2\alpha_2} \\ &=& (a_1e_1 + a_2e_2)E_{-(\alpha_1+2\alpha_2)} + (-b_1d_1 - b_2d_2)E_{\alpha_1+2\alpha_2} \subset \mathfrak{m}_{2\alpha_2}; \end{array}$$

$$3)\,[Y,Z] = c_1e_1E_{-\alpha_2} + c_1e_2E_{-(\alpha_1+\alpha_2)} - c_2d_1E_{\alpha_2} - c_2d_2E_{\alpha_1+\alpha_2} \subset \mathfrak{m}_{\alpha_2}.$$

$\square$

Theorem 4. *Consider the flag manifold $G_2(\alpha_2)$ and $V \in \mathfrak{m}$. Write $V = X + Y + Z$, where X, Y, Z are as in (7). Then the equation $[V, \Lambda V]_{\mathfrak{m}} = 0$, for any invariant metric Λ, is equivalent to the following system of algebraic equations*

$$\left\{ \begin{array}{rcl} b_2c_1 &=& 0 \\ c_2d_1 &=& 0 \\ b_1c_1 &=& 0 \\ c_2d_2 &=& 0 \\ b_1d_1 + b_2d_2 &=& 0 \\ a_1c_1 &=& 0 \\ a_2c_1 &=& 0 \end{array} \right. . \tag{8}$$

Therefore V is an equigeodesic vector if, and only if, one of the following holds:
a) $V \in \mathfrak{m}_{\alpha_2}$;
b) $V \in \mathfrak{m}_{2\alpha_2}$;
c) $V \in \mathfrak{m}_{3\alpha_2}$;
d) $V \in \mathfrak{u}_{\alpha_1+\alpha_2} \oplus \mathfrak{u}_{\alpha_1+3\alpha_2}$;
e) $c_1 = 0, c_2 = 0, d_1 = -\frac{b2*d2}{b1}$.

Proof: Let $V = X + Y + Z \in \mathfrak{m}$, where X, Y, Z are as in equation (7). We have:

$$\begin{array}{rcl} [V, \Lambda V]_{\mathfrak{m}} &=& [X + Y + Z, \lambda_1 X + \lambda_2 Y + \lambda_3 Z] \\ &=& (\lambda_2 - \lambda_1)[X,Y] + (\lambda_3 - \lambda_1)[X,Z] + (\lambda_3 - \lambda_2)[Y,Z] \\ &=& (\lambda_2 - \lambda_1)\{a_1c_1E_{\alpha_1+3\alpha_2} + a_1c_2E_{-(\alpha_1+\alpha_2)} + a_2c_1E_{2\alpha_1+3\alpha_2} + a_2c_2E_{-\alpha_2} \\ && -b_1c_1E_{\alpha_1+\alpha_2} - b_1c_2E_{-(\alpha_1+3\alpha_2)} - b_2c_1E_{\alpha_2} - b_2c_2E_{-(2\alpha_1+3\alpha_2)}\} \\ && +(\lambda_3 - \lambda_1)\{(a_1e_1 + a_2e_2)E_{-(\alpha_1+2\alpha_2)} + (-b_1d_1 - b_2d_2)E_{\alpha_1+2\alpha_2}\} \\ && +(\lambda_3 - \lambda_2)\{c_1e_1E_{-\alpha_2} + c_1e_2E_{-(\alpha_1+\alpha_2)} - c_2d_1E_{\alpha_2} - c_2d_2E_{\alpha_1+\alpha_2}\}. \end{array}$$

Then V is an equigeodesic vector if, and only if, the coefficients of V satisfy the system of equations (8).

The solutions of (8) are:

a) $c_1 = 0, c_2 = 0, d_1 = -\frac{b2*d2}{b1}$;
b) $c_1 = 0, d_1 = 0, d_2 = 0$;
c) $b_1 = 0, c_1 = 0, c_2 = 0, d_2 = 0$;
d) $a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0, c_2 = 0$;
e) $a_1 = 0, a_2 = 0, b_1 = 0, b_2 = 0, d_1 = 0, d_2 = 0$;
f) $b_1 = 0, b_2 = 0, c_1 = 0, c_2 = 0$.

Therefore the solutions of the algebraic system (8) determine all vector spaces that appear in the Theorem 4.

□

6 Rank three T-Roots and Einstein Metrics

In the study of the geometry of flag manifolds a class of metrics play a key role namely the Einstein metrics. We recall that an invariant metric Λ on $\mathbb{F}$ is Einstein if it is proportional to the Ricci tensor, that is, it satisfies $Ric_\Lambda = c\Lambda$.

It is well known that the problem of finding invariant Einstein metrics on flag manifolds reduces to solve a delicate algebraic system, [23]. A well known solution for this system is the Kähler-Einstein one. Indeed, for each invariant complex structure on $\mathbb{F}$ there exist a unique invariant Kähler-Einstein metric, see ([4], 8.95).

There are several examples of invariant Einstein non-Kähler metrics. All of them have repetition in the parameters of the metric, see [12] or [21]. In this section we connect this repetition on the parameters of an invariant Einstein metric with the dimension of the isotropic summands $\mathfrak{m}_i$.

We derive it using the expression of the scalar curvature correspondent to an invariant metric and our result concerning zero sum triple of T-roots. We prove that there are no isolated T-root, that is, every T-root belongs to a zero sum triple of T-roots.

Let $\langle \cdot, \cdot \rangle$ be the Cartan-Killing form on $\mathfrak{g}$ and

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s$$

a decomposition into irreducible (non-equivalent) $\mathrm{ad}(\mathfrak{k})-$submodules. Let $\{X_\alpha\}$ be a orthonormal basis (with respect to $-\langle \cdot, \cdot \rangle$) adapted to the decomposition of $\mathfrak{m}$: $X_\alpha \in \mathfrak{m}_i$ and $X_\beta \in \mathfrak{m}_j$ with $i < j$ then $\alpha < \beta$. Following [23] we set $A^\gamma_{\alpha\beta} := -\langle [X_\alpha, X_\beta], X_\gamma \rangle$, thus $[X_\alpha, X_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} X_\gamma$. Consider

$$C^k_{ij} := \sum (A^\gamma_{\alpha\beta})^2 \tag{9}$$

where the sum is taken over all indexes α, β, γ with $X_\alpha \in \mathfrak{m}_i, X_\beta \in \mathfrak{m}_j, X_\gamma \in \mathfrak{m}_k$. Hence C^k_{ij} is nonnegative, symmetric in all of the three entries, and is independent of the orthonormal basis chosen for $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$ (but it depends on the choice of the decomposition of $\mathfrak{m}$).

Let Λ be an invariant metric on $\mathbb{F}$ and $S(\Lambda)$ the correspondent scalar curvature. According to [23]

$$S(\Lambda) = \frac{1}{2} \sum_i \frac{D_i}{\lambda_i} - \frac{1}{4} \sum_{i,j,k} C^k_{ij} \frac{\lambda_k}{\lambda_i \lambda_j} \tag{10}$$

where $D_i = \dim_{\mathbb{R}}(\mathfrak{m}_i)$ and λ_i denotes the parameter of the invariant metric Λ with $i = 1, \ldots, s$. We consider now the set of the invariant metrics with unitary volume:

$$\mathcal{M} = \{(\lambda_1, \ldots, \lambda_s) \in \mathbb{R}^s : \lambda_1^{D_1} \cdots \lambda_s^{D_s} = 1;\ \lambda_1, \ldots, \lambda_s > 0\}.$$

The next result is in fact true for any compact, connected homogeneous space. It shows an alternative manner of computing the Einstein equations.

Theorem 5. *([4]) Let $\mathbb{F}$ be a flag manifold. Then the critical points of the restriction map $S|_{\mathcal{M}}$ are precisely the invariant Einstein metrics on $\mathbb{F}$.*

Lemma 3. *([1]) Let ξ, η, ζ be T-roots such that $\xi + \eta + \zeta = 0$. Then there exist roots $\alpha, \beta, \gamma \in R$ with $k(\alpha) = \xi$, $k(\beta) = \eta$, $k(\gamma) = \zeta$, such that $\alpha + \beta + \gamma = 0$.*

The calculus of the coefficients C_{ij}^k can be laborius. However the next result shows exactly which of them are nonzero.

Lemma 4. *Let $\delta_i, \delta_j, \delta_k$ T-roots associated to the real $ad(\mathfrak{k})$-modules $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$, respectively . Then $C_{ij}^k \neq 0$ if, and only if, $\delta_i + \delta_j + \delta_k = 0$.*

Proof: Consider the vectors E_α, $\alpha \in R_M$, of the fixed Weyl basis of $\mathfrak{g}^{\mathbb{C}}$, and $V_\alpha := \mathbb{R}S_\alpha + \mathbb{R}\sqrt{-1}A_\alpha$, $\alpha \in R_M^+$. Hence, the vectors $I_\alpha = S_\alpha/\sqrt{2}$ and $F_\alpha = \sqrt{-1}A_\alpha/\sqrt{2}$, $\alpha \in R_M^+$ form a orthonormal basis of V_α. Thus, each set $b_i = \left\{I_\alpha, F_\alpha : k(\alpha) = \delta_i, \alpha \in R_M^+\right\}$ is a orthonormal basis of a $ad(\mathfrak{k})$-module $\mathfrak{m}_i = \mathfrak{m}_{\delta_i}$, with $\delta_i \in R_T^+$.

Let b_i, b_j and b_k be a orthonormal basis of $\mathfrak{m}_i$, $\mathfrak{m}_j$ and $\mathfrak{m}_k$, respectively. We notice that $\left[\mathfrak{g}_\alpha^{\mathbb{C}}, \mathfrak{g}_\beta^{\mathbb{C}}\right] = \mathfrak{g}_{\alpha+\beta}^{\mathbb{C}}$ and $\left(\mathfrak{g}_{\alpha+\beta}^{\mathbb{C}}, \mathfrak{g}_\gamma^{\mathbb{C}}\right) = 0$, unless $\alpha + \beta + \gamma = 0$. Then $-\langle [e_\alpha, e_\beta], e_\gamma\rangle = 0$ except when $\alpha + \beta + (-\gamma) = 0$, for any $e_\alpha \in b_i$, $e_\beta \in b_j$ and $e_\gamma \in b_k$, with $\alpha, \beta, \gamma \in R_M^+$.

If $C_{ij}^k \neq 0$ then exists $\alpha, \beta, \gamma \in R_M^+$ with $k(\alpha) = \delta_i$, $k(\beta) = \delta_j$, $k(\gamma) = \delta_k$ such that $\alpha + \beta + (-\gamma) = 0$. Hence $\delta_i + \delta_j + (-\delta_k) = k(\alpha) + k(\beta) + (-k(\gamma)) = k(\alpha + \beta + (-\gamma)) = 0$.

Conversely, if $\delta_i, \delta_j, \delta_k$ are nonzero T-roots such that $\delta_i + \delta_j + \delta_k = 0$, then there exist $\alpha, \beta, \gamma \in R_M$ with $k(\alpha) = \delta_i$, $k(\beta) = \delta_j$, $k(\gamma) = \delta_k$ such that $\alpha + \beta + \gamma = 0$, hence $C_{ij}^k \neq 0$. □

Definition 2. *Let $\mathbb{F}$ be a flag manifold and R_T the correspondent set of T-roots We say that a T-root δ_i* belongs to a T-zero sum triple *if there are T-roots $\delta_j, \delta_k \in R_T$ such that $\delta_i + \delta_j + \delta_k = 0$. In this case we denote by $T(\delta_i)$ the number of T-zero sum triple contained the T-root δ_i. Of course $T(\delta_i) = T(-\delta_i)$, for every $\delta_i \in R_T$.*

We now connect the repetition on the parameters of an invariant Einstein metric with the dimension of its associated isotropic summands.

Proposition 4. *Let δ_i, δ_j, $\delta_k \in R_T$ with δ_i, δ_j, $\delta_k = -(\delta_i + \delta_j)$ such that $T(\delta_i) = T(\delta_j) = 1$. If there exists i and j such that $\mathbb{F}$ admits an invariant Einstein metric Λ satisfying $\lambda_i = \lambda_j$ then* $\dim \mathfrak{m}_i = \dim \mathfrak{m}_j$.

Proof: According Theorem 5 an invariant metric Λ is Einstein if, and only if, Λ is solution of the $s+1$ equations

$$\frac{\partial S}{\partial \lambda_l} = \xi D_l \lambda_1^{D_1} \cdots \lambda_l^{D_l - 1} \cdots \lambda_s^{D_s},\ 1 \leq l \leq s \tag{11}$$

$$\lambda_1^{D_1} \cdots \lambda_s^{D_s} = 1 \tag{12}$$

where $D_l = \dim \mathfrak{m}_l$ and ξ denotes the Lagrange multiplier.

In particular, Λ must satisfy the two equations of (11) for $l = i$ and $l = j$. By hypothesis $T(\delta_i) = T(\delta_j) = 1$, then using Lemma 9 and formula (10) we conclude that the equation (11), for $l = i$, reduces to

$$-\frac{D_i}{2\lambda_i^2} - \frac{1}{4} C_{ij}^k \left(\frac{1}{\lambda_j \lambda_k} - \frac{\lambda_k}{\lambda_i^2 \lambda_j} - \frac{\lambda_j}{\lambda_i^2 \lambda_k}\right) = \xi D_i \lambda_1^{D_1} \cdots \lambda_i^{D_i - 1} \cdots \lambda_s^{D_s}. \tag{13}$$

Multiplying this equality by λ_i/D_i and using (12) we obtain

$$-\frac{1}{2\lambda_i} - \frac{1}{4D_i} C_{ij}^k \left(\frac{\lambda_i}{\lambda_j \lambda_k} - \frac{\lambda_k}{\lambda_i \lambda_j} - \frac{\lambda_j}{\lambda_i \lambda_k}\right) = \xi. \tag{14}$$

Analogously, if $l = j$ we obtain

$$-\frac{1}{2\lambda_j} - \frac{1}{4D_j} C_{ij}^k \left(\frac{\lambda_j}{\lambda_i \lambda_k} - \frac{\lambda_k}{\lambda_i \lambda_j} - \frac{\lambda_i}{\lambda_j \lambda_k} \right) = \xi. \tag{15}$$

Therefore if Λ is an invariant Einstein metric satisfying $\lambda_i = \lambda_j$, according to equations (14) and (15), we obtain $D_i = D_j$. □

The next result shows that the set of T-roots enjoys an interesting property, despite not being a root system.

Theorem 6. *Let $\mathbb{F}$ be a flag manifold and R_T a correspondent set of T-roots. If R_T contain more than one positive T-root, then every T-root belongs to some T-zero sum triple.*

Proof: Consider the invariant Kähler-Einstein metric Λ_K on $\mathbb{F}$ associated to the canonical invariant complex structure on $\mathbb{F}$. Λ_K satisfies the algebraic system (11) and (12).

Suppose the existence of a T-root δ_{k_0} which do not belong to any T-zero sum triple. According to Lemma 4, we see that $C_{ij}^{k_0} = 0$ for every $i, j = 1, \ldots, s$. Thus, the equation (11) for $l = k_0$ reduces to

$$-\frac{D_{k_0}}{2\lambda_{k_0}^2} = \xi D_{k_0} \lambda_1^{D_1} \cdots \lambda_{k_0}^{D_{k_0}-1} \cdots \lambda_s^{D_s}$$

then

$$\xi = -\frac{1}{2\lambda_{k_0}}. \tag{16}$$

By assumption there exist a positive T-root $\delta_i \neq \delta_{k_0}$. Then the equation (11) for $l = i$ becomes

$$\frac{\partial S}{\partial \lambda_i} = -\frac{1}{2\lambda_{k_0}} D_i \lambda_1^{D_1} \cdots \lambda_i^{D_i - 1} \cdots \lambda_s^{D_s}.$$

Then,

$$\frac{\partial S}{\partial \lambda_i} = -\frac{D_i}{2\lambda_i \lambda_{k_0}}. \tag{17}$$

On the other hand,

$$\frac{\partial S}{\partial \lambda_i} = -\frac{D_i}{2\lambda_i^2} + f \tag{18}$$

, f being a function depending on each λ_j whose T-roots satisfies $\delta_i \pm \delta_j \in R_T$ (according to 4). Therefore using (17) and (18) we see that

$$f = \frac{D_i}{2\lambda_i} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_{k_0}} \right).$$

This equality contradicts the fact that δ_{k_0} do not belongs to any T-zero sum triple. This concludes the proof. □

This result allow us to characterize some sets of T-roots.

Corollary 2. *Let $\mathbb{F}$ be a flag manifold such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, then $R_T = \{\pm\zeta, \pm 2\zeta\}$ where $\zeta \in \mathfrak{t}^* \setminus 0$. Also, if $\dim \mathfrak{m}_1 \neq \dim \mathfrak{m}_2$ then any invariant Einstein metric on $\mathbb{F}$ satisfies $\lambda_1 \neq \lambda_2$.*

Proof: As $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ we may write the set of T-roots in the form $R_T = \{\delta, -\delta, \zeta, -\zeta\}$ with $\delta, \zeta \in k(R^+)$. By Theorem 6, δ belongs to some T-zero sum triple. But, T-roots are nonzero linear functionals in $\mathfrak{t}^*$, then the possibilities for the T-zero sum triple containing δ are $\delta + \zeta + \zeta = 0$, $\delta + \delta + \zeta = 0$ and $\delta - \zeta - \zeta = 0$. According to Lemma 3, we see that the first and the second possibilities cannot happens because $\delta, \zeta \in k(R^+)$, then $\delta = 2\zeta$.

For the remaining possibility we see that each T-root belongs to exactly one T-zero sum triple. Now Proposition 4 follows from the Corollary. □

In the case of three isotropic summands (irreducibles and inequivalent) some T-roots may belong to more than one T-zero sum triple, but even in this case we can see that there exist few possibilities for the set of T-roots.

Corollary 3. *Let $\mathbb{F}$ be a flag manifold such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, then $R_T = \{\pm\delta, \pm\zeta, \pm(\delta+\zeta)\}$ or $R_T = \{\pm\delta, \pm 2\delta, \pm 4\delta\}$, where $\delta, \zeta \in \mathfrak{t}^* \setminus \{0\}$.*

Proof: By assumption, the set of T-roots is given by $R_T = \{\pm\alpha, \pm\beta, \pm\gamma\}$ where $\alpha, \beta, \gamma \in k(R^+)$ with $\alpha, \beta, \gamma \in \mathfrak{t}^* \setminus \{0\}$.

But do not exist T-zero sum triple with only positive T-roots or containing two opposite sign T-roots. Thus, the possibilities for the T-zero sum triples containing α are $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, \gamma)$, $(\alpha, -\beta, -\gamma)$, $(\alpha, \alpha, -\beta)$, $(\alpha, \alpha, -\gamma)$, $(\alpha, -\beta, -\beta)$, $(\alpha, -\gamma, -\gamma)$.

For any of the first three choices we conclude that the set of T-roots has the form $R_T = \{\pm\delta, \pm\zeta, \pm(\delta+\zeta)\}$. For the last four choices we obtain that the set of T-roots has the form $R_T = \{\pm\delta, \pm 2\delta, \pm 3\delta\}$ or $R_T = \{\pm\delta, \pm 2\delta, \pm 4\delta\}$. □

Example 4. *Consider the flag manifold $G_2(\alpha_2)$.It is a direct computation to verify that in this case $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with $\dim \mathfrak{m}_1 = 8$ and $\dim \mathfrak{m}_2 = 2$. Hence, any invariant Einstein metric has exactly two parameters λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$, according to Corollary 3.* □

Example 5. *Consider the flag manifold $\mathbb{F}$ =$G_2(\alpha_1)$. In this case the subalgebra $\mathfrak{t}$ has the form*

$$diag\{\delta, \delta, -2\delta\} \in \mathfrak{su}(3)$$

and a choice of positive T-roots is the same as a choice of a set of linear functionals of the form $R_T^+ = \{\delta, 2\delta, 3\delta\}$. Then $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ where $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$ correspond to the functionals $\delta, 2\delta$ and 3δ, respectively.

It is easy to check that the T-roots δ and 2δ belong to exactly one T-zero sum triple and $\dim \mathfrak{m}_1 = 4$, $\dim \mathfrak{m}_2 = 2$. Then any invariant Einstein metric on $\mathbb{F}$ has exactly three parameters λ_1, λ_2 and λ_3 satisfying $\lambda_1 \neq \lambda_2$. As $\dim \mathfrak{m}_3 = 4$ and $T(3\delta) = 1$. In a similar way, we prove that any invariant Einstein metric must satisfy $\lambda_1 \neq \lambda_3$. This gives an alternative proof of the well known fact that the normal metric is not Einstein on $G_2/U(2)$. □

We, just for completeness, conclude this section determining the Einstein equations for the full flag manifold G_2/T and derive explicitly, the invariant Kähler-Einstein metric corresponding to each invariant complex structure.

In the case of maximal flag manifolds, the irreducible and inequivalent isotropic summands $\mathfrak{m}_i$ are determined by the positive roots R^+. So they are indexed by these roots and for the coefficients C_{ij}^k we may write $C_{\alpha\beta}^{\gamma}$ with $\alpha, \beta, \gamma \in R^+$.

According to [19] if $\mathbb{F} = G/T$ is a full flag manifold then the Einstein equation of a invariant metric g on $\mathbb{F}$ is given by

$$c = \frac{1}{2\lambda_\alpha} + \frac{1}{8} \sum_{\beta,\gamma \in R^+} \frac{\lambda_\alpha}{\lambda_\beta \lambda_\gamma} C_{\beta\gamma}^{\alpha} - \frac{1}{4} \sum_{\beta,\gamma \in R^+} \frac{\lambda_\gamma}{\lambda_\alpha \lambda_\beta} C_{\alpha\beta}^{\gamma} \tag{19}$$

where $g(\cdot,\cdot) = \lambda_\alpha(\cdot,\cdot)|_{\mathfrak{m}_\alpha}$, $\alpha \in R^+$, c is the Einstein constant and T is a maximal torus on G.

Now, using the fixed Weyl base of $\mathfrak{g}^{\mathbb{C}}$ we see that the unique triple of positive roots in $\mathfrak{g}^{\mathbb{C}}$ such that $C_{\alpha\beta}^{\gamma}$ is nonzero are

$$C_{\alpha\beta}^{\alpha+\beta} = 2\left(N_{\alpha,\beta}\right)^2 \qquad \text{and} \qquad C_{\alpha\beta}^{\alpha-\beta} = 2\left(N_{\alpha,-\beta}\right)^2 \tag{20}$$

where $N_{\alpha,\beta}$ are the constants of structure of $\mathfrak{g}^{\mathbb{C}}$. Thus, we obtain the Einstein equation for G_2/T.

Proposition 5. *The Einstein equations for the full flag manifold $\mathbb{F} = G_2/T$ are given by the following algebraic system*

$$
\begin{aligned}
c &= \frac{1}{2\lambda_{\alpha_1}} + \frac{1}{16}\left(\frac{\lambda_{\alpha_1}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+3\alpha_2}}\right) - \frac{1}{16}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_1+\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{\alpha_1}\lambda_{2\alpha_1+3\alpha_2}} + \frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_1+3\alpha_2}}\right)\\
c &= \frac{1}{2\lambda_{\alpha_2}} + \frac{1}{16}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_1}} + \frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right) + \frac{1}{12}\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_1+\alpha_2}} - \frac{1}{12}\left(\frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_2}\lambda_{\alpha_1+2\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_2}\lambda_{\alpha_1+\alpha_2}}\right)\\
&\quad -\frac{1}{16}\left(\frac{\lambda_{\alpha_1}}{\lambda_{\alpha_2}\lambda_{\alpha_1+\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_2}\lambda_{\alpha_1}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_2}\lambda_{\alpha_1+3\alpha_2}} + \frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right)\\
c &= \frac{1}{2\lambda_{\alpha_1+\alpha_2}} + \frac{1}{16}\left(\frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right) + \frac{1}{12}\frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_2}} - \frac{1}{12}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_1+2\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_2}}\right)\\
&\quad -\frac{1}{16}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_1}} + \frac{\lambda_{\alpha_1}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{2\alpha_1+3\alpha_2}} + \frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right)\\
c &= \frac{1}{2\lambda_{\alpha_1+2\alpha_2}} + \frac{1}{16}\left(\frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+\alpha_2}}\right) + \frac{1}{12}\frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_2}} - \frac{1}{12}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_1+\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_2}}\right)\\
&\quad -\frac{1}{16}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_1+3\alpha_2}} + \frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{2\alpha_1+3\alpha_2}} + \frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_1+\alpha_2}}\right)\\
c &= \frac{1}{2\lambda_{\alpha_1+3\alpha_2}} - \frac{1}{16}\left(\frac{\lambda_{\alpha_2}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{\alpha_1+2\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{2\alpha_1+3\alpha_2}} + \frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+3\alpha_2}\lambda_{\alpha_1}}\right)\\
&\quad +\frac{1}{16}\left(\frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+2\alpha_2}\lambda_{\alpha_2}} + \frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1}}\right)\\
c &= \frac{1}{2\lambda_{2\alpha_1+3\alpha_2}} - \frac{1}{16}\left(\frac{\lambda_{\alpha_1+3\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1}} + \frac{\lambda_{\alpha_1}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+3\alpha_2}} + \frac{\lambda_{\alpha_1+2\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+\alpha_2}} + \frac{\lambda_{\alpha_1+\alpha_2}}{\lambda_{2\alpha_1+3\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right)\\
&\quad +\frac{1}{16}\left(\frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1}\lambda_{\alpha_1+3\alpha_2}} + \frac{\lambda_{2\alpha_1+3\alpha_2}}{\lambda_{\alpha_1+\alpha_2}\lambda_{\alpha_1+2\alpha_2}}\right).
\end{aligned}
$$

Proof: Using (20) we compute $C_{\alpha\beta}^{\gamma}$, with $\alpha, \beta, \gamma \in \mathbb{R}^+$. Now the result follows from equation (19). □

Let $\mathbb{F}$ be a flag manifold with a invariant complex structure J fixed. We recall that there exists a bijection between partial ordering in R_M and complex structures on $\mathbb{F}$. This correspondence is given by

$$JE_\alpha = \pm iE_\alpha \qquad \alpha \in R_M^+.$$

It is well known that for each invariant complex structure (or equivalently, partial ordering in R_M) there exist a unique (up to homotheties) invariant Kähler-Einstein metric, see ([4],Chapter 8) or ([19]). This metric is given by

$$\Lambda_J = \{\lambda_\alpha = (\delta, \alpha) : \delta = \frac{1}{2}\sum_{\beta \in R_M^+} \beta\} \tag{21}$$

where $(\cdot,\cdot)$ is a inner product on $\mathfrak{h}^*$ induced by the Cartan-Killing form of $\mathfrak{g}$.

According to [5] given a invariant complex structure J on $\mathbb{F}$, we have a simple criterion satisfied for invariant Kähler metrics on flags: a invariant metric Λ is Kähler (with respect to J) if and only if

$$\lambda_{\alpha+\beta} = \lambda_\alpha + \lambda_\beta \qquad \alpha, \beta \in R_M^+. \tag{22}$$

But, for each Weyl chamber of the usual root system of $\mathfrak{g}_2$ we have a choice of positive roots. As this root system has twelve roots and they form successive angles of 30^o, there are exactly twelve Weyl chambers, then we have twelve possible invariant complex structure or six pairs of conjugate structures.

On the other hand, if an invariant metric Λ is Kähler with respect to J, then Λ is also Kähler with respect to $-J$. So it sufficient to consider only the non conjugate invariant complex structure to describe

all the invariant Kähler-Einstein metrics. For each invariant complex structure we have a choice of positive roots.

Now we describe explicitly the (unique) invariant Kahler-Einstein metric associated to each complex structure on G_2/T.

Proposition 6. *The maximal flag manifold G_2/T admits exactly (up to homotheties) six invariant Kähler-Einstein metrics. These metrics and the correspond choice of positive roots are given in the following table.*

Λ	R^+	Σ
$(3,1,4,5,6,9)$	$\alpha_2, \alpha_1+3\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_1$	α_1, α_2
$(6,5,1,4,9,3)$	$-(\alpha_1+\alpha_2), -\alpha_1, \alpha_2, \alpha_1+3\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2$	$-(\alpha_1+\alpha_2), 2\alpha_1+3\alpha_2$
$(3,4,1,5,9,6)$	$-\alpha_1, \alpha_2, \alpha_1+3\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2, \alpha_1+\alpha_2$	$-\alpha_1, \alpha_1+\alpha_2$
$(6,1,5,4,3,9)$	$\alpha_1+3\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_1, -\alpha_2$	$\alpha_1+3\alpha_2, -\alpha_2$
$(9,4,5,1,3,6)$	$\alpha_1+2\alpha_2, 2\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_1, -\alpha_2, -(\alpha_1+3\alpha_2)$	$\alpha_1+2\alpha_2, -(\alpha_1+3\alpha_2)$
$(9,5,4,1,6,3)$	$2\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_1, -\alpha_2, -(\alpha_1+3\alpha_2), -(\alpha_1+2\alpha_2)$	$2\alpha_1+3\alpha_2, -(\alpha_1+2\alpha_2)$

where in the first column $\Lambda = (\lambda_{\alpha_1}, \lambda_{\alpha_2}, \lambda_{\alpha_1+\alpha_2}, \lambda_{\alpha_1+2\alpha_2}, \lambda_{\alpha_1+3\alpha_2}, \lambda_{2\alpha_1+3\alpha_2})$.

Proof: The proof is obtained using the formula (21). We perform the canonical choice for positives roots (corresponding to the first row of the above table). The proof for the other rows in the table are done in similar way. The canonical choice for the roots is $R^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, 2\alpha_1+3\alpha_2\}$.

Consider the fundamental weights Λ_1, Λ_2 related to simple roots α_1, α_2 and defined by $\frac{2(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$, $i,j = 1,2$. Using the Cartan matrix of $\mathfrak{g}_2$ we can write the simple roots in terms of the fundamental weights, see [14]. The Cartan matrix of $\mathfrak{g}_2$ is given by

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

and therefore $\alpha_1 = 2\Lambda_1 - 3\Lambda_2$ and $\alpha_2 = -\Lambda_1 + 2\Lambda_2$. Then,

$$\begin{aligned} 2\delta &= \alpha_1+\alpha_2+\alpha_1+\alpha_2+\alpha_1+2\alpha_1+\alpha_1+3\alpha_2+2\alpha_1+3\alpha_2 \\ &= 6\alpha_1+10\alpha_2 \\ &= 6(2\Lambda_1-3\Lambda_2)+10(-\Lambda_1+2\Lambda_2) \\ &= 2\Lambda_1+2\Lambda_2, \end{aligned}$$

and $\delta = \Lambda_1+\Lambda_2$. But $\frac{2(\Lambda_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$ and setting $(\alpha_1,\alpha_1) = (\alpha_1+3\alpha_2, \alpha_1+3\alpha_2) = (2\alpha_1+3\alpha_2, 2\alpha_1+3\alpha_2) = 3$ (for long roots) and $(\alpha_2,\alpha_2) = (\alpha_1+\alpha_2, \alpha_1+\alpha_2) = (\alpha_1+2\alpha_2, \alpha_1+2\alpha_2) = 1$ (for short roots) we have

$$\begin{aligned} \lambda_{\alpha_1} &= (\Lambda_1+\Lambda_2, \alpha_1) = \frac{1}{2}(\alpha_1,\alpha_1) = \frac{3}{2} \\ \lambda_{\alpha_2} &= (\Lambda_1+\Lambda_2, \alpha_2) = (\Lambda_2,\alpha_2) = \frac{1}{2}(\alpha_2,\alpha_2) = \frac{1}{2} \\ \lambda_{\alpha_1+\alpha_2} &= (\Lambda_1+\Lambda_2, \alpha_1+\alpha_2) = (\Lambda_1,\alpha_1)+(\Lambda_2,\alpha_2) = 2 \\ \lambda_{\alpha_1+2\alpha_2} &= (\Lambda_1+\Lambda_2, \alpha_1+2\alpha_2) = (\Lambda_1,\alpha_1)+2(\Lambda_2,\alpha_2) = \frac{5}{2} \\ \lambda_{\alpha_1+3\alpha_2} &= (\Lambda_1+\Lambda_2, \alpha_1+3\alpha_2) = (\Lambda_1,\alpha_1)+3(\Lambda_2,\alpha_2) = 3 \\ \lambda_{2\alpha_1+3\alpha_2} &= (\Lambda_1+\Lambda_2, 2\alpha_1+3\alpha_2) = 2(\Lambda_1,\alpha_1)+3(\Lambda_2,\alpha_2) = \frac{9}{2}. \end{aligned}$$

Thus an invariant Kähler-Einstein metric on G_2/T is given (up to scale) by $\Lambda = (3/2, 1/2, 2, 5/2, 3, 9/2)$, and after normalization, we obtain the metric in the table. □

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